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Approximation of semigroups of Lipschitz operators

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ABSTRACT

This paper is devoted to an approximation theorem of semigroups of Lipschitz operators which are closely related with the abstract Cauchy problem and an application of the obtained result to the projection method for the two-dimensional Navier–Stokes equations.

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1. Introduction

Let X be a real Banach space with norm $\|\cdot\|$ and let D be a closed set in X. Our purpose is to obtain an approximation theorem of semigroups of Lipschitz operators which are closely related with the abstract Cauchy problem in X

$$(\mathrm{ACP}; A, x) \quad u'(t) \in Au(t) \quad \text{for } t > 0, \quad \text{and} \quad u(0) = x \in D$$

and to apply the obtained result to the projection method for the two-dimensional Navier–Stokes equations. A one-parameter family $\{S(t); t \geq 0\}$ of Lipschitz operators from D into itself is called a *semigroup of Lipschitz operators* on D if the following three conditions are satisfied:

(S1) $S(0)x = x$ for $x \in D$, and $S(t+s)x = S(t)S(s)x$ for $s, t \geq 0$ and $x \in D$.
(S2) For $x \in D$, $S(\cdot)x : [0, \infty) \to X$ is continuous.
(S3) For $\tau > 0$ there exists $M_\tau > 0$ such that

$$\|S(t)x - S(t)y\| \leq M_\tau \|x - y\| \quad \text{for } x, y \in D \text{ and } t \in [0, \tau].$$

If $M_\tau = 1$ for $\tau > 0$, then a semigroup $\{S(t); t \geq 0\}$ on D is said to be *contractive*. The problem of approximating contractive semigroups by discrete semigroups has been studied intensively in [1–6].

An attempt to extend such a problem to the case of semigroups of Lipschitz operators was made in [7] by proposing a stability condition of discrete semigroups in the following sense: let $\{D_h; h \in (0, h_0]\}$ be a family of subsets of X such that $D \subset D_h$ for $h \in (0, h_0]$ and let C_h be a Lipschitz operator from D_h into itself for $h \in (0, h_0]$. Then the family $\{C_h; h \in (0, h_0]\}$ is said to be *stable* if for $\tau > 0$ there exists $M_\tau > 0$ such that $\|C_h^n x - C_h^n y\| \leq M_\tau \|x - y\|$ for $x, y \in D_h$, $h \in (0, h_0]$ and $n \geq 1$ with $nh \in [0, \tau]$. It is known in [7,8] that this condition is equivalent to the existence of a family $\{\Phi_h; h \in (0, h_0]\}$ of nonnegative functionals on $X \times X$ satisfying $(\Phi 1)$ and $(\Phi 2)$ in the next section such that $\Phi_h(C_h x, C_h y) \leq e^{\omega h}\Phi_h(x, y)$

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for $x, y \in D_h$ and $h \in (0, h_0]$. A product formula for semigroups under such a stability condition and certain consistency condition different from our condition (C2) was obtained in [9,10].

A generation theorem of semigroups of Lipschitz operators discussed in [11] was applied to Navier–Stokes equations by showing a dissipativity condition in terms of a functional of the form $\Phi(v, w) = N_v(v - w)$, where $\{N_v; v \in D\}$ is a family of equivalent norms in X which varies Lipschitz continuously in v. In order to introduce dissipativity conditions, such a family of norms was also used in [12,13]. These considerations lead us to the use of a family $\{\Phi_h; h \in (0, h_0]\}$ of nonnegative functionals on $X \times X$ satisfying (Φ3) in addition to (Φ1) and (Φ2).

The main theorem is given by Theorem 2.2, which is an extension of [3, Theorem 2.2]. The feature in our setting is that C_h is assumed to be multi-valued for $h \in (0, h_0]$ in order to apply to *Projection Method* described in [14, Chapter III, Section 7] and that the family $\{C_h; h \in (0, h_0]\}$ is defined to be stable if for any regular data $u \in D_h$ there exists a nice approximation $w^h \in C_h u$ satisfying (2.1). Section 3 contains an application to the projection method for the two-dimensional Navier–Stokes equations. Here we would like to express our appreciation to the referees for suggesting how best to improve our original manuscript.

2. Approximation of semigroups of Lipschitz operators

Let $h_0 > 0$ and let $\{D_h; h \in (0, h_0]\}$ be a family of subsets of X. Let $\{\Phi_h; h \in (0, h_0]\}$ be a family of nonnegative functionals on $X \times X$ satisfying the following three conditions:

(Φ1) There exists $L > 0$ such that

$$|\Phi_h(x, y) - \Phi_h(\hat{x}, \hat{y})| \le L(\|x - \hat{x}\| + \|y - \hat{y}\|)$$

for $(x, y), (\hat{x}, \hat{y}) \in D_h \times D_h$ and $h \in (0, h_0]$.

(Φ2) There exist $M \ge m > 0$ such that

$$m\|x - y\| \le \Phi_h(x, y) \le M\|x - y\| \quad \text{for } (x, y) \in D_h \times D_h \text{ and } h \in (0, h_0].$$

(Φ3) For $v \in D_h$ and $h \in (0, h_0]$ the functional $X \ni w \to \Phi_h(v, w)$ is convex and Lipschitz continuous with Lipschitz constant L_0 independent of v and h.

Let $\{C_h; h \in (0, h_0]\}$ be a family of multi-valued operators in X such that $D(C_h) = D_h$ for $h \in (0, h_0]$ and $C_h u \subset D_h$ for $u \in D_h$ and $h \in (0, h_0]$.

Assume that there exist $a, b, \omega \ge 0$ and a family $\{\varphi_h; h \in (0, h_0]\}$ of proper functionals from X into $[0, \infty]$ such that $D(\varphi_h) \subset D_h$ for $h \in (0, h_0]$, where $D(\varphi_h)$ is the effective domain of φ_h, and the following stability condition (C1) and consistency condition (C2) are satisfied:

(C1) For $\alpha > 0$ there exists $h_\alpha \in (0, h_0]$ such that to each $h \in (0, h_\alpha]$ and $u \in D(\varphi_h)$ satisfying $\varphi_h(u) \le \alpha$ there corresponds $w^h \in C_h u$ such that

$$\Phi_h(w^h, \hat{w}) \le e^{\omega h}\Phi_h(u, \hat{u}) \quad \text{for } \hat{u} \in D_h \text{ and } \hat{w} \in C_h\hat{u}, \tag{2.1}$$

$$\varphi_h(w^h) \le e^{ah}(\varphi_h(u) + bh). \tag{2.2}$$

(C2) For any $u \in D(A)$ and $\xi \in Au$ there exist $u^h \in D(\varphi_h)$ and $\xi^h \in A_h u^h$ for $h \in (0, h_0]$ such that $\lim_{h\to 0+} u^h = u$, $\lim_{h\to 0+} \xi^h = \xi$ and $\limsup_{h\to 0+} \varphi_h(u^h) < \infty$, where

$$A_h = h^{-1}(C_h - I) \quad \text{for } h \in (0, h_0].$$

Remark 2.1. If $D = \overline{D(A)}$, then for any $u \in D$ there exist $u^h \in D_h$ for $h \in (0, h_0]$ such that $\lim_{h\to 0+} u^h = u$.

Theorem 2.2. *Assume that $D = \overline{D(A)}$. Let $\{S(t); t \ge 0\}$ be a semigroup of Lipschitz operators on D such that for $\tau > 0$ and $x \in D$, $u(t) := S(t)x|_{[0,\tau]}$ gives a mild solution to (ACP; A, x) on $[0, \tau]$. Then for any $x \in D$ and $v_0^h \in D_h$ for $h \in (0, h_0]$ such that $\lim_{h\to 0+} v_0^h = x$,*

$$S(t)x = \lim_{h\to 0+, jh\to t} v_j^h \quad \text{in } X \tag{2.3}$$

provided that $v_j^h \in D_h$ for $j \ge 1$ are defined by $v_j^h \in C_h v_{j-1}^h$ for $j \ge 1$. Here the convergence is uniform on any bounded subinterval of $[0, \infty)$.

Proof. Let $x_0 \in D$ and let $\{v_0^h; h \in (0, h_0]\}$ be a family in X such that $v_0^h \in D_h$ for $h \in (0, h_0]$ and $\lim_{h\to 0+} v_0^h = x_0$ in X. For $h \in (0, h_0]$ let $\{v_j^h\}_{j=1}^\infty$ be a sequence in X such that $v_j^h \in D_h$ and $v_j^h \in C_h v_{j-1}^h$ for $j \ge 1$.

Let $\bar{\tau} > 0$ and let $u_0 \in D(A)$ and $\xi_0 \in Au_0$ be arbitrary. By condition (C2) there exist $u_0^h \in D(\varphi_h)$ and $\xi_0^h \in A_h u_0^h$ for $h \in (0, h_0]$ such that $\lim_{h\to 0+} u_0^h = u_0$, $\lim_{h\to 0+} \xi_0^h = \xi_0$ and $\limsup_{h\to 0+} \varphi_h(u_0^h) < \infty$. Since $\limsup_{h\to 0+} \varphi_h(u_0^h) < \infty$, there exist $\bar{\alpha} > 0$ and $\bar{h} \in (0, h_0]$ such that $\varphi_h(u_0^h) \le \bar{\alpha}$ for $h \in (0, \bar{h}]$. Let $\alpha = e^{a\bar{\tau}}(\bar{\alpha} + b\bar{\tau})$. Then, by condition (C1) there exists $h_\alpha \in (0, \bar{h}]$ satisfying the following condition: for $h \in (0, h_\alpha]$ and $u \in D(\varphi_h)$ with $\varphi_h(u) \le \alpha$ there exists $w^h \in C_h u$ such that $\Phi_h(w^h, \hat{w}) \le e^{\omega h}\Phi_h(u, \hat{u})$ for $\hat{u} \in D_h$ and $\hat{w} \in C_h\hat{u}$ and such that $\varphi_h(w^h) \le e^{ah}(\varphi_h(u) + bh)$. This condition is denoted by (*).

Let $h \in (0, h_\alpha]$ and let K be an integer such that $hK \le \bar{\tau}$. Then there exists a sequence $\{u_j^h\}_{j=0}^K$ in X such that $u_j^h \in D(\varphi_h)$, $u_j^h \in C_h u_{j-1}^h$,

$$\Phi_h(u_j^h, \hat{w}) \le e^{\omega h}\Phi_h(u_{j-1}^h, \hat{u}) \quad \text{for } \hat{u} \in D_h \text{ and } \hat{w} \in C_h\hat{u}, \tag{2.4}$$

$$\varphi_h(u_j^h) \le e^{ajh}(\varphi_h(u_0^h) + bjh) \tag{2.5}$$

for $1 \le j \le K$. Moreover, we have

$$\Phi_h(u_j^h, v_j^h) \le e^{\omega jh}\Phi_h(u_0^h, v_0^h) \quad \text{for } 0 \le j \le K. \tag{2.6}$$

Indeed, let $1 \le j \le K$ and assume that a sequence $\{u_k^h\}_{k=0}^{j-1}$ in X has been chosen, as required. Since

$$\varphi_h(u_{j-1}^h) \le e^{a(j-1)h}(\varphi_h(u_0^h) + b(j-1)h), \tag{2.7}$$

we have $\varphi_h(u_{j-1}^h) \le e^{a\bar{\tau}}(\bar{\alpha}+b\bar{\tau}) = \alpha$. Therefore, we apply condition (*) with $u = u_{j-1}^h \in D(\varphi_h)$ to find an element $u_j^h \in C_h u_{j-1}^h$ satisfying (2.4) and

$$\varphi_h(u_j^h) \le e^{ah}(\varphi_h(u_{j-1}^h) + bh).$$

This inequality combined with (2.7) implies that $\varphi_h(u_j^h) \le e^{ah}(e^{a(j-1)h}(\varphi_h(u_0^h) + b(j-1)h) + bh) \le a^{ajh}(\varphi_h(u_0^h) + bjh)$. Thus, the desired sequence $\{u_j^h\}_{j=0}^K$ can be inductively chosen. Since $v_{j-1}^h \in D_h$ and $v_j^h \in C_h v_{j-1}^h$ for $j \ge 1$, by (2.4) we have $\Phi_h(u_j^h, v_j^h) \le e^{\omega h}\Phi(u_{j-1}^h, v_{j-1}^h)$ for $1 \le j \le K$. The desired inequality (2.6) is obtained by solving this recursive inequality.

Let $0 < \tau < \bar{\tau}$ and let $u : [0, \tau] \to X$ be a mild solution to (ACP; A, x_0) on $[0, \tau]$. Then, there exists a family $\{u^\lambda; \lambda > 0\}$ of λ-approximate solutions such that $\|u^\lambda(t) - u(t)\| \le \lambda$ for $t \in [0, \tau]$. By the definition of λ-approximate solutions, for each $\lambda > 0$ there exist $x_0^\lambda \in X$ and $(t_k^\lambda, x_k^\lambda, z_k^\lambda) \in [0, \infty) \times D(A) \times X$ for $1 \le k \le N^\lambda$ such that $0 = t_0^\lambda < t_1^\lambda < \cdots < t_{N^\lambda-1}^\lambda < \tau \le t_{N^\lambda}^\lambda$, $h_k^\lambda := t_k^\lambda - t_{k-1}^\lambda \le \lambda$ for $1 \le k \le N^\lambda$, $(x_k^\lambda - x_{k-1}^\lambda)/h_k^\lambda \in Ax_k^\lambda + z_k^\lambda$ for $1 \le k \le N^\lambda$, $\sum_{k=1}^{N^\lambda} h_k^\lambda \|z_k^\lambda\| \le \lambda$ and

$$u^\lambda(t) = \begin{cases} x_0^\lambda & \text{for } t = 0, \\ x_k^\lambda & \text{for } t \in (t_{k-1}^\lambda, t_k^\lambda] \cap [0, \tau] \text{ and } 1 \le k \le N^\lambda. \end{cases}$$

Let $\xi_k^\lambda = (x_k^\lambda - x_{k-1}^\lambda)/h_k^\lambda - z_k^\lambda$ for $1 \le k \le N^\lambda$ and $\lambda > 0$. Since $x_k^\lambda \in D(A)$ and $\xi_k^\lambda \in Ax_k^\lambda$ for $1 \le k \le N^\lambda$ and $\lambda > 0$, we deduce from condition (C2) that for $1 \le k \le N^\lambda$ and $\lambda > 0$ there exist $x_k^{\lambda,h} \in D(\varphi_h)$ and $\xi_k^{\lambda,h} \in A_h x_k^{\lambda,h}$ for $h \in (0, h_0]$ such that $x_k^{\lambda,h} \to x_k^\lambda$ and $\xi_k^{\lambda,h} \to \xi_k^\lambda$ as $h \to 0+$.

Let $\bar{\lambda}_0 > 0$ be a number satisfying $\bar{\lambda}_0\omega < 1/2$ and choose $\bar{h}_0 \in (0, h_\alpha]$ such that $\omega_h := h^{-1}(e^{\omega h} - 1) \le 1/2\bar{\lambda}_0$ for $h \in (0, \bar{h}_0]$. Then we have

$$0 \le \lambda\omega_h \le 1/2 \quad \text{for } h \in (0, \bar{h}_0] \text{ and } \lambda \in (0, \bar{\lambda}_0]. \tag{2.8}$$

Let $\lambda \in (0, \bar{\lambda}_0]$ and $h \in (0, \bar{h}_0]$. Condition (Φ3) with $v = u_j^h \in D_h$ for $0 \le j \le K$ implies that the function $t \to \Phi_h(u_j^h, x_k^{\lambda,h} + t\xi_k^{\lambda,h})$ is convex for $0 \le j \le K$ and $1 \le k \le N^\lambda$. This yields that $(\Phi_h(u_j^h, x_k^{\lambda,h}) - \Phi_h(u_j^h, x_k^{\lambda,h} - h_k^\lambda\xi_k^{\lambda,h}))/h_k^\lambda \le (\Phi_h(u_j^h, x_k^{\lambda,h} + h\xi_k^{\lambda,h}) - \Phi_h(u_j^h, x_k^{\lambda,h}))/h$ for $0 \le j \le K$ and $1 \le k \le N^\lambda$. Since the functional $X \ni w \to \Phi_h(u_j^h, w)$ is Lipschitz continuous with Lipschitz constant L_0 for $0 \le j \le K$, we have

$$(\Phi_h(u_j^h, x_k^{\lambda,h}) - \Phi_h(u_j^h, x_{k-1}^{\lambda,h}))/h_k^\lambda \le (\Phi_h(u_j^h, x_k^{\lambda,h} + h\xi_k^{\lambda,h}) - \Phi_h(u_j^h, x_k^{\lambda,h}))/h + L_0\|z_k^{\lambda,h}\| \tag{2.9}$$

for $0 \le j \le K$ and $1 \le k \le N^\lambda$, where

$$x_0^{\lambda,h} = v_0^h \quad \text{and} \quad z_k^{\lambda,h} = (x_k^{\lambda,h} - x_{k-1}^{\lambda,h})/h_k^\lambda - \xi_k^{\lambda,h} \quad \text{for } 1 \le k \le N^\lambda.$$

Since $x_k^{\lambda,h} + h\xi_k^{\lambda,h} \in C_h x_k^{\lambda,h}$ for $1 \le k \le N^\lambda$, by (2.4) we have

$$\Phi_h(u_j^h, x_k^{\lambda,h} + h\xi_k^{\lambda,h}) \le e^{\omega h}\Phi_h(u_{j-1}^h, x_k^{\lambda,h}) \tag{2.10}$$

for $1 \le j \le K$ and $1 \le k \le N^\lambda$. Substituting (2.10) into (2.9) we find that

$$\Phi_h(u_j^h, x_k^{\lambda,h}) \le \frac{h}{h + h_k^\lambda}\Phi_h(u_j^h, x_{k-1}^{\lambda,h}) + \frac{h_k^\lambda}{h + h_k^\lambda}e^{\omega h}\Phi_h(u_{j-1}^h, x_k^{\lambda,h}) + \frac{hh_k^\lambda}{h + h_k^\lambda}L_0\|z_k^{\lambda,h}\| \tag{2.11}$$

for $1 \le j \le K$ and $1 \le k \le N^\lambda$. Let $\gamma_k^{\lambda,h} = \prod_{l=1}^k (1 - h_l^\lambda \omega_h)$ for $0 \le k \le N^\lambda$. Then we want to show that

$$e^{-\omega hj}\gamma_k^{\lambda,h}\Phi_h(u_j^h, x_k^{\lambda,h}) \le \Phi_h(u_0^h, x_0^{\lambda,h}) + L_0\sum_{l=1}^k h_l^\lambda \|z_l^{\lambda,h}\| + L(M/m)((jh - t_k^\lambda)^2 + jh^2 + \lambda t_k^\lambda)^{1/2}\|\xi_0^h\| \tag{2.12}$$

for $0 \le j \le K$ and $0 \le k \le N^\lambda$. Since $u_0^h + h\xi_0^h \in C_h u_0^h \subset D_h$, by (2.4) with $j = 1$ we have $\Phi_h(u_1^h, u_0^h + h\xi_0^h) \le 0$; hence $(u_1^h - u_0^h)/h = \xi_0^h$ by condition (Φ2). By (2.4) we have $\Phi_h(u_{j+1}^h, u_j^h) \le e^{\omega h}\Phi_h(u_j^h, u_{j-1}^h)$ for $1 \le j \le K-1$, which yields that $\Phi_h(u_j^h, u_{j-1}^h) \le e^{\omega(j-1)h}\Phi_h(u_1^h, u_0^h)$ for $1 \le j \le K$. By condition (Φ2) this inequality implies that

$$m\|u_j^h - u_{j-1}^h\| \le Me^{\omega(j-1)h}\|u_1^h - u_0^h\| \le hMe^{\omega(j-1)h}\|\xi_0^h\| \tag{2.13}$$

for $1 \le j \le K$. By condition (Φ1) we have $|\Phi_h(u_j^h, x_0^{\lambda,h}) - \Phi_h(u_{j-1}^h, x_0^{\lambda,h})| \le L\|u_j^h - u_{j-1}^h\|$ for $1 \le j \le K$. This together with (2.13) yields that $\Phi_h(u_j^h, x_0^{\lambda,h}) - \Phi_h(u_{j-1}^h, x_0^{\lambda,h}) \le L(M/m)e^{\omega(j-1)h}h\|\xi_0^h\|$ for $1 \le j \le K$. Solving this recursive inequality we observe that the inequality (2.12) is true for $0 \le j \le K$ and $k = 0$. Since $x_k^{\lambda,h} \in D(\varphi_h) \subset D_h$ and $\xi_k^{\lambda,h} \in A_h x_k^{\lambda,h}$, we have $x_k^{\lambda,h} + h\xi_\lambda^{\lambda,h} \in C_h x_k^{\lambda,h} \subset D_h$ for $1 \le k \le N^\lambda$. Since $h^{-1}|\Phi_h(u_0^h, x_k^{\lambda,h} + h\xi_k^{\lambda,h}) - \Phi_h(u_1^h, x_k^{\lambda,h} + h\xi_k^{\lambda,h})| \le L\|\xi_0^h\|$, we use (2.9) with $j = 0$ and (2.10) with $j = 1$ to find that

$$(\Phi_h(u_0^h, x_k^{\lambda,h}) - \Phi_h(u_0, x_{k-1}^{\lambda,h}))/h_k^\lambda \le \omega_h\Phi_h(u_0^h, x_k^{\lambda,h}) + L_0\|z_k^{\lambda,h}\| + L\|\xi_0^h\|$$

for $1 \le k \le N^\lambda$. Since $\gamma_{k-1}^{\lambda,h} \le 1$ and $M/m \ge 1$, we have

$$\gamma_k^{\lambda,h}\Phi_h(u_0^h, x_k^{\lambda,h}) \le \gamma_{k-1}^{\lambda,h}\Phi_h(u_0^h, x_{k-1}^{\lambda,h}) + L_0 h_k^\lambda\|z_k^{\lambda,h}\| + L(M/m)h_k^\lambda\|\xi_0^h\|$$

for $1 \le k \le N^\lambda$. This recursive inequality implies that the inequality (2.12) holds for $j = 0$ and $0 \le k \le N^\lambda$. From (2.11) we can inductively show that the inequality (2.12) is true for $0 \le j \le K$ and $0 \le k \le N^\lambda$ (see also [3, Lemma 3.3]).

In order to prove (2.3), let $t \in [0, \tau]$. Then there exists an integer $1 \le k \le N^\lambda$ such that $t \in (t_{k-1}^\lambda, t_k^\lambda]$ or $t = t_0^\lambda$. Let $0 < \delta < \bar\tau - \tau$ be arbitrary and let $|jh - t| \le \delta$. Then we notice that $0 \le j \le K$. Since $\|v_j^h - u(t)\| \le \|v_j^h - u_j^h\| + \|u_j^h - x_k^{\lambda,h}\| + \|x_k^{\lambda,h} - x_k^\lambda\| + \|u^\lambda(t) - u(t)\|$ and since $\lim_{h\to 0+}\max_{1\le k\le N^\lambda}\|x_k^{\lambda,h} - x_k^\lambda\| = 0$ for $\lambda > 0$, we find by (2.6) and (2.12) that

$$\begin{aligned}&\limsup_{\delta\to 0+}(\sup\{\|v_j^h - u(t)\|;\ |jh - t| \le \delta, 0 < h \le \delta, t \in [0, \tau]\})\\ &\quad \le (M/m)e^{\omega\bar\tau}\|u_0 - x_0\| + e^{\omega\bar\tau + 2\omega(\tau+\lambda)}\{M\|u_0 - x_0\| + L(M/m)(\lambda^2 + \lambda(\tau+\lambda))^{1/2}\|\xi_0\| + L_0\lambda\}/m + \lambda.\end{aligned}$$

Here we have used the inequality

$$(1-t)^{-1} \le e^{2t} \quad \text{for } 0 \le t \le 1/2 \tag{2.14}$$

and (2.8) to obtain $(\gamma_k^{\lambda,h})^{-1} \le e^{2t_k^\lambda\omega_h}$ for $0 \le k \le N^\lambda$. Since $D(A)$ is dense in D, we obtain (2.3) by letting $\lambda \to 0+$ and $u_0 \to x_0$. $\square$

3. Projection method for Navier–Stokes equations

This section is devoted to the projection method for the Navier–Stokes equation

$$\text{(NS)}\ \begin{cases} u_t + (u\cdot\nabla)u - \nu\Delta u + \nabla p = 0, \quad \operatorname{div} u = 0 & \text{in } \Omega\times(0,\infty),\\ u = 0 & \text{on } \partial\Omega\times(0,\infty),\end{cases}$$

where the unknown function $u = u(x,t)$ is a vector-valued function from $\Omega\times(0,\infty)$ into $\mathbb{R}^2$, Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and $\nu > 0$.

Let $\boldsymbol{L}^2(\Omega) = L^2(\Omega)^2$ and $\langle u, v\rangle = \sum_{i=1}^2\int_\Omega u_i v_i\,dx$ for $u, v \in \boldsymbol{L}^2(\Omega)$. Let $\boldsymbol{H}_0^1(\Omega) = H_0^1(\Omega)^2$ and $\langle\langle u, v\rangle\rangle = \sum_{i=1}^2\langle\nabla u_i, \nabla v_i\rangle$ for $u, v \in \boldsymbol{H}_0^1(\Omega)$, and define

$$\hat b(u, v, w) = (b(u, v, w) - b(u, w, v))/2 \quad \text{for } u, v, w \in \boldsymbol{H}_0^1(\Omega),$$

where $b(u, v, w) = \langle(u\cdot\nabla)v, w\rangle$ for $u, v, w \in \boldsymbol{H}_0^1(\Omega)$. Then we have

$$\hat b(u, v, v) = 0 \quad \text{for } u, v \in \boldsymbol{H}_0^1(\Omega). \tag{3.1}$$

Moreover, we use the following inequality in [14]:

$$|b(u, v, w)| \le \sqrt{2}|u|^{1/2}\|u\|^{1/2}\|v\|\,|w|^{1/2}\|w\|^{1/2} \quad \text{for } u, v, w \in \boldsymbol{H}_0^1(\Omega), \tag{3.2}$$

where $|z| = \sqrt{\langle z, z\rangle}$ for $z \in \boldsymbol{L}^2(\Omega)$ and $\|z\| = \sqrt{\langle\langle z, z\rangle\rangle}$ for $z \in \boldsymbol{H}_0^1(\Omega)$. Let $v \in \boldsymbol{H}_0^1(\Omega)$ and let $\{v_n\}$ be a sequence in $\boldsymbol{H}_0^1(\Omega)$ such that $v_n \to v$ in $\boldsymbol{L}^2(\Omega)$ and $v_n \to v$ weakly in $\boldsymbol{H}_0^1(\Omega)$ as $n \to \infty$. Then, by (3.2) we have $\lim_{n\to\infty} \hat{b}(v_n - v, v_n, \phi) = 0$ for $\phi \in \boldsymbol{H}_0^1(\Omega)$. Since for $u, w \in \boldsymbol{H}_0^1(\Omega)$, the functional $v \to \hat{b}(u, v, w)$ is linear and bounded on $\boldsymbol{H}_0^1(\Omega)$, we have $\lim_{n\to\infty} \hat{b}(v, v_n - v, \phi) = 0$ for $\phi \in \boldsymbol{H}_0^1(\Omega)$. Therefore, we observe that for any $v \in \boldsymbol{H}_0^1(\Omega)$ and any sequence $\{v_n\}$ in $\boldsymbol{H}_0^1(\Omega)$,

$$\lim_{n\to\infty} \hat{b}(v_n, v_n, \phi) = \hat{b}(v, v, \phi) \quad \text{for } \phi \in \boldsymbol{H}_0^1(\Omega) \tag{3.3}$$

provided that $v_n \to v$ in $\boldsymbol{L}^2(\Omega)$ and $v_n \to v$ weakly in $\boldsymbol{H}_0^1(\Omega)$ as $n \to \infty$.

Let X be the closure of the set $\{u \in C_0^\infty(\Omega)^2;\ \operatorname{div} u = 0\}$ in $\boldsymbol{L}^2(\Omega)$. Let $u^0 \in X$ and $h > 0$. If u^m is defined for $m \geq 0$, then $u^{m+1/2} \in \boldsymbol{H}_0^1(\Omega)$ and $u^{m+1} \in X$ are defined in the following way:

$$\langle (u^{m+1/2} - u^m)/h, \phi\rangle + \nu\langle\langle u^{m+1/2}, \phi\rangle\rangle + \hat{b}(u^{m+1/2}, u^{m+1/2}, \phi) = 0 \quad \text{for } \phi \in \boldsymbol{H}_0^1(\Omega), \tag{3.4}$$

$$\langle u^{m+1}, \phi\rangle = \langle u^{m+1/2}, \phi\rangle \quad \text{for } \phi \in X. \tag{3.5}$$

It is proved by the Galerkin procedure that for $u \in X$ and $h > 0$ there exists $v \in \boldsymbol{H}_0^1(\Omega)$ such that $\langle (v-u)/h, \phi\rangle + \nu\langle\langle v, \phi\rangle\rangle + \hat{b}(v, v, \phi) = 0$ for $\phi \in \boldsymbol{H}_0^1(\Omega)$. This ensures the existence of at least one element $u^{m+1/2}$ satisfying (3.4). The element u^{m+1} defined by $u^{m+1} = P_X u^{m+1/2}$ satisfies (3.5), where P_X denotes the orthogonal projection in $\boldsymbol{L}^2(\Omega)$ on the space X. The way to find a solution through the limit of the sequence $\{u^m\}$ is called the *Projection Method* (see [14]).

Theorem 3.1 *([14, Theorem 3.7.1]). Let u be a weak solution of* (NS) *. Let $u^0 \in X$ and $h > 0$ and let $\{u^m\}$ be the sequence defined by (3.4) and (3.5). Then for $t \geq 0$,*

$$u(t) = \lim_{h\to 0+, mh\to t} u^m \quad \text{in } X. \tag{3.6}$$

Let V be the closure of the set $\{u \in C_0^\infty(\Omega)^2;\ \operatorname{div} u = 0\}$ in $\boldsymbol{H}_0^1(\Omega)$. Let $u^0 \in X$. Let $r \geq |u^0|^2$ and let $D = \{u \in X;\ |u|^2 \leq r\}$. It was proved in [11] that a semigroup $\{S(t);\ t \geq 0\}$ of Lipschitz operators on D is generated by the operator A in X defined in such a way that $Av = \xi$ if and only if $v \in D \cap V$, $\xi \in X$ and

$$\langle \xi, \phi\rangle + \nu\langle\langle v, \phi\rangle\rangle + b(v, v, \phi) = 0 \quad \text{for } \phi \in V \tag{3.7}$$

and that the semigroup $\{S(t);\ t \geq 0\}$ gives a weak solution of (NS).

Let $r_0 > r$ and let $D_h = \{u \in X;\ |u|^2 \leq r_0\}$ for $h > 0$. Then, it is obvious that $D \subset D_h$ for $h > 0$. Consider the family $\{C_h;\ h > 0\}$ of multi-valued operators in X defined in such a way that $w \in C_h u$ if and only if $u \in X$, $w \in X$, $|u|^2 \leq r_0$ and there exists $v \in \boldsymbol{H}_0^1(\Omega)$ such that

$$\langle (v - u)/h,\ \phi\rangle + \nu\langle\langle v, \phi\rangle\rangle + \hat{b}(v, v, \phi) = 0 \quad \text{for } \phi \in \boldsymbol{H}_0^1(\Omega), \tag{3.8}$$

$$\langle w, \phi\rangle = \langle v, \phi\rangle \quad \text{for } \phi \in X. \tag{3.9}$$

It is clear that $D(C_h) = D_h$ for $h > 0$. Let $h > 0$, $u \in D_h$ and $w \in C_h u$. Then, by the definition of C_h we have $u \in X$, $w \in X$, $|u|^2 \leq r_0$ and there exists $v \in \boldsymbol{H}_0^1(\Omega)$ satisfying (3.8) and (3.9). Setting $\phi = v \in \boldsymbol{H}_0^1(\Omega)$ in (3.8) we have, by (3.1), $(|v|^2 - |u|^2)/h + 2\nu\|v\|^2 \leq 0$, which implies that $|v|^2 \leq |u|^2 \leq r_0$. By (3.9) we have $|w|^2 \leq |v|^2$; hence $w \in D_h$. This proves that $C_h u \subset D_h$ for $u \in D_h$ and $h > 0$ and that the sequence $\{u^m\}$ defined by (3.4) and (3.5) satisfies $u^m \in C_h u^{m-1}$ for $m \geq 1$. We shall apply Theorem 2.2 to demonstrate that a weak solution u is obtained through the formula (3.6). To verify (C1), we need the following lemma.

Lemma 3.2. *For $h > 0$ and $u \in D_h$ there exists $w \in C_h u$ such that*

$$|w - u| = \inf_{z\in C_h u} |z - u|.$$

Proof. Let $d = \inf_{z\in C_h u} |z - u|$. Then there exists $z_n \in C_h u$ for $n \geq 1$ such that $\lim_{n\to\infty} |z_n - u| = d$. By the definition of C_h we have $u \in X$, $z_n \in X$ for $n \geq 1$, $|u|^2 \leq r_0$ and there exists a sequence $\{v_n\}$ in $\boldsymbol{H}_0^1(\Omega)$ such that

$$\langle (v_n - u)/h, \phi\rangle + \nu\langle\langle v_n, \phi\rangle\rangle + \hat{b}(v_n, v_n, \phi) = 0 \quad \text{for } \phi \in \boldsymbol{H}_0^1(\Omega) \text{ and } n \geq 1, \tag{3.10}$$

$$\langle z_n, \phi\rangle = \langle v_n, \phi\rangle \quad \text{for } \phi \in X \text{ and } n \geq 1. \tag{3.11}$$

Since the sequence $\{z_n\}$ is bounded in X, there exist $w \in X$ and a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \to w$ weakly in X as $k \to \infty$. Setting $\phi = v_n \in \boldsymbol{H}_0^1(\Omega)$ in (3.10), we find by (3.1) that $|v_n|^2 + 2h\nu\|v_n\|^2 \leq |u|^2$ for $n \geq 1$. This implies that the sequence $\{v_n\}$ is bounded in $\boldsymbol{H}_0^1(\Omega)$. Therefore, there exist $v \in \boldsymbol{H}_0^1(\Omega)$ and a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $v_{n_k} \to v$ weakly in $\boldsymbol{H}_0^1(\Omega)$ as $k \to \infty$. Since $\boldsymbol{H}_0^1(\Omega)$ is compactly embedded in $\boldsymbol{L}^2(\Omega)$, we have $v_{n_k} \to v$ in $\boldsymbol{L}^2(\Omega)$ as $k \to \infty$. By (3.3) we have $\lim_{k\to\infty} \hat{b}(v_{n_k}, v_{n_k}, \phi) = \hat{b}(v, v, \phi)$ for $\phi \in \boldsymbol{H}_0^1(\Omega)$. A passage to the limit in (3.10) and (3.11) with $n = n_k$ yields that $w \in C_h u$ and $d \leq |w - u|$. Since $|w - u| \leq \liminf_{k\to\infty} |z_{n_k} - u| = d$, we have $|w - u| = d$. $\square$

By (3.2) we have $|\hat{b}(w,z,w)| \le (\sqrt{2}/2)(|w|\|w\|\|z\| + |w|^{1/2}\|w\|^{3/2}|z|^{1/2}\|z\|^{1/2})$ for $w, z \in \boldsymbol{H}_0^1(\Omega)$ and an application of Young's inequality yields that the right-hand side is bounded by $(\sqrt{2}/2)(\epsilon\|w\|^2 + C(\epsilon)\|z\|^2|w|^2 + \epsilon\|w\|^2 + C(\epsilon)|w|^2|z|^2\|z\|^2)$ for any $\epsilon > 0$. Therefore, for any $\epsilon > 0$ there exists $c_\epsilon > 0$ such that

$$|\hat{b}(w,z,w)| \le \epsilon\|w\|^2 + c_\epsilon(1+|z|^2)\|z\|^2|w|^2 \quad \text{for } w, z \in \boldsymbol{H}_0^1(\Omega). \tag{3.12}$$

Let $\{\Phi_h; h > 0\}$ be the family of nonnegative functionals on $X \times X$ defined by

$$\Phi_h(u, \hat{u}) = \exp(\nu^{-1}c_\nu(1+r_0)|u|^2)|u-\hat{u}|$$

for $(u, \hat{u}) \in X \times X$ and $h > 0$, where c_ν is a constant satisfying (3.12) with $\epsilon = \nu$. Then, conditions $(\Phi 1)$ through $(\Phi 3)$ are clearly satisfied. Let $\{\varphi_h; h > 0\}$ be the family of proper functionals from X into $[0, \infty]$ defined by

$$\varphi_h(u) = \begin{cases} \exp(\nu^{-1}c_\nu(1+r_0)|u|^2)|||A_h u||| & \text{if } u \in D_h, \\ \infty & \text{otherwise,} \end{cases}$$

where $|||A_h u||| = \inf\{|\xi|; \xi \in A_h u\}$ for $u \in D_h$ and $h > 0$.

Lemma 3.3. *Let $h > 0$, $u \in D(\varphi_h)$ and $\varphi_h(u) \le \alpha$. Assume that $w \in C_h u$ satisfies that $|(w-u)/h| = |||A_h u|||$. Let $\hat{u} \in D_h$ and $\hat{w} \in C_h \hat{u}$. Then*

$$\exp(\nu^{-1}c_\nu(1+r_0)|w|^2)|w-\hat{w}| \le \exp(\nu^{-1}c_\nu(1+r_0)|u|^2)|u-\hat{u}|$$

if $hc_\nu(1+r_0)\nu^{-1}\alpha\sqrt{r_0} \le 1/2$.

Proof. Let $h > 0$, $u \in D(\varphi_h)$ and $\varphi_h(u) \le \alpha$ and assume that $hc_\nu(1+r_0)\nu^{-1}\alpha\sqrt{r_0} \le 1/2$. Since $w \in C_h u$, we have $u \in X$, $w \in X$, $|u|^2 \le r_0$ and there exists $v \in \boldsymbol{H}_0^1(\Omega)$ satisfying (3.8) and (3.9). Setting $\phi = v \in \boldsymbol{H}_0^1(\Omega)$ in (3.8), we have $|v-u|^2/h + \nu\|v\|^2 = -\langle (v-u)/h, u\rangle = -\langle (w-u)/h, u\rangle$, where we have used (3.9) to obtain the last equality. Since $|(w-u)/h| = |||A_h u||| \le \varphi_h(u) \le \alpha$, we have

$$\nu\|v\|^2 \le |||A_h u||| \cdot |u| \le \alpha\sqrt{r_0}. \tag{3.13}$$

Since $\hat{w} \in C_h\hat{u}$, there exists $\hat{v} \in \boldsymbol{H}_0^1(\Omega)$ such that

$$\langle (\hat{v}-\hat{u})/h, \phi\rangle + \nu\langle\langle \hat{v}, \phi\rangle\rangle + \hat{b}(\hat{v}, \hat{v}, \phi) = 0 \quad \text{for } \phi \in \boldsymbol{H}_0^1(\Omega), \tag{3.14}$$

$$\langle \hat{w}, \phi\rangle = \langle \hat{v}, \phi\rangle \quad \text{for } \phi \in X. \tag{3.15}$$

Subtracting (3.14) from (3.8), setting $\phi = v - \hat{v} \in \boldsymbol{H}_0^1(\Omega)$ in the resulting equality and using the inequality $\langle w - z, w\rangle \ge |w|(|w| - |z|)$ for $w, z \in \boldsymbol{L}^2(\Omega)$, we find that

$$|v-\hat{v}|(|v-\hat{v}| - |u-\hat{u}|)/h + \nu\|v-\hat{v}\|^2 + \hat{b}(v, v, v-\hat{v}) - \hat{b}(\hat{v}, \hat{v}, v-\hat{v}) \le 0.$$

By (3.1) and (3.12) we have $|\hat{b}(v,v,v-\hat{v}) - \hat{b}(\hat{v},\hat{v},v-\hat{v})| = |\hat{b}(v-\hat{v}, v, v-\hat{v})| \le \nu\|v-\hat{v}\|^2 + c_\nu(1+|v|^2)\|v\|^2|v-\hat{v}|^2$. Setting $\phi = v \in \boldsymbol{H}_0^1(\Omega)$ in (3.8), we have

$$(|v|^2 - |u|^2)/h + 2\nu\|v\|^2 \le 0, \tag{3.16}$$

which implies that $|v|^2 \le |u|^2 \le r_0$. These inequalities together imply that

$$(|v-\hat{v}| - |u-\hat{u}|)/h \le c_\nu(1+r_0)\|v\|^2|v-\hat{v}|. \tag{3.17}$$

Notice by (3.13) that $hc_\nu(1+r_0)\|v\|^2 \le hc_\nu(1+r_0)\nu^{-1}\alpha\sqrt{r_0} \le 1/2$. Then, by (2.14) we have $|v-\hat{v}| \le \exp(2hc_\nu(1+r_0)\|v\|^2)|u-\hat{u}|$. Substituting (3.16) into this inequality, we obtain the desired inequality, since $|w-\hat{w}| \le |v-\hat{v}|$ and $|w|^2 \le |v|^2$ by (3.9) and (3.15). $\square$

To verify (C1), let $\alpha > 0$ and choose $h_\alpha > 0$ such that $h_\alpha c_\nu(1+r_0)\nu^{-1}\alpha\sqrt{r_0} \le 1/2$. Let $h \in (0, h_\alpha]$, $u \in D(\varphi_h)$ and $\varphi_h(u) \le \alpha$. Then, Lemma 3.2 asserts that there exists $w^h \in C_h u$ such that $|(w^h - u)/h| = |||A_h u|||$. Let $\hat{u} \in D_h$ and $\hat{w} \in C_h\hat{u}$. Then we deduce from Lemma 3.3 that (2.1) is satisfied with $\omega = 0$. To check (2.2), let $f^h \in C_h w^h$. By Lemma 3.3 we have $\exp(\nu^{-1}c_\nu(1+r_0)|w^h|^2)|w^h - f^h| \le \exp(\nu^{-1}c_\nu(1+r_0)|u|^2)|u-w^h|$. Since $(f^h - w^h)/h \in A_h w^h$, we have $|||A_h w^h||| \le |(f^h - w^h)/h|$. Therefore, we obtain (2.2) with $a = b = 0$, and the verification of (C1) is completed. To verify (C2) we need the following two lemmas.

Lemma 3.4. *Let $u \in D(A)$. Then there exists $\lambda_0 > 0$ such that if $\hat{u} \in V$ satisfies*

$$\langle (\hat{u}-u)/\lambda_0 + Au, \phi\rangle + \nu\langle\langle \hat{u}, \phi\rangle\rangle + b(\hat{u}, \hat{u}, \phi) = 0 \quad \textit{for } \phi \in V, \tag{3.18}$$

then $u = \hat{u}$.

Proof. By (3.7) we have $\langle Au, \phi\rangle + \nu\langle\langle u, \phi\rangle\rangle + b(u, u, \phi) = 0$ for $\phi \in V$. Substituting this into (3.18) and setting $\phi = \hat{u} - u \in V$ in the resulting equality, we find that $|\hat{u} - u|^2 + \lambda_0\nu\|\hat{u} - u\|^2 = -\lambda_0 b(\hat{u} - u, u, \hat{u} - u)$, since $b(w, z, z) = 0$ for $w, z \in V$. By (3.2), the right-hand side is estimated by $\lambda_0(\nu\|\hat{u} - u\|^2 + (1/2\nu)|\hat{u} - u|^2\|u\|^2)$, where we have used the Young inequality. If $\lambda_0 > 0$ is chosen such that $(\lambda_0/2\nu)\|u\|^2 < 1$, then $\hat{u} = u$. □

Lemma 3.5. *Let $\lambda > 0$, $h > 0$ and $f \in X$. Then there exist $\bar{u} \in X$ and $\bar{v} \in \boldsymbol{H}_0^1(\Omega)$ such that*

$$\langle(\bar{v} - \bar{u})/h, \phi\rangle + \nu\langle\langle\bar{v}, \phi\rangle\rangle + \hat{b}(\bar{v}, \bar{v}, \phi) = 0 \quad \text{for } \phi \in \boldsymbol{H}_0^1(\Omega), \tag{3.19}$$

$$\bar{u} = \frac{h}{\lambda + h} f + \frac{\lambda}{\lambda + h} P_X \bar{v}. \tag{3.20}$$

Moreover, $|\bar{v}| \le |\bar{u}| \le |f|$.

Proof. Let $E = \boldsymbol{H}_0^1(\Omega)$. Consider the mapping T from E into E^* defined by

$$\langle T(w), \phi\rangle_{E^*,E} = \left\langle w - \frac{\lambda}{\lambda + h} P_X w, \phi\right\rangle + h\nu\langle\langle w, \phi\rangle\rangle + h\hat{b}(w, w, \phi)$$

for $w, \phi \in E$ and the functional F in E^* defined by $\langle F, \phi\rangle_{E^*,E} = \langle \frac{h}{\lambda+h} f, \phi\rangle$ for $\phi \in E$. Then, the first assertion can be proved if $\bar{v} \in E$ is found such that $T(\bar{v}) = F$ and $\bar{u} \in X$ is defined by (3.20). We want to show that $R(T) = E^*$. By (3.2) we observe that T is bounded. By (3.1) we have

$$\langle T(w), w\rangle_{E^*,E} = |w|^2 - \frac{\lambda}{\lambda + h}\langle P_X w, w\rangle + h\nu\|w\|^2 \ge \frac{h}{\lambda + h}|w|^2 + h\nu\|w\|^2$$

for $w \in E$, where we have used the fact that $\|P_X\| \le 1$ to obtain the last inequality. This means that T is coercive. To show that T is pseudo-monotone, let $w \in E$ and let $\{w_n\}$ be a sequence in E such that it converges weakly in E to w as $n \to \infty$ and such that $\limsup_{n\to\infty}\langle T(w_n), w_n - w\rangle_{E^*,E} \le 0$. Since E is compactly embedded in $\boldsymbol{L}^2(\Omega)$, we have $w_n \to w$ in $\boldsymbol{L}^2(\Omega)$ as $n \to \infty$. From (3.1) and (3.3) we deduce that $\hat{b}(w_n, w_n, w_n - w) = \hat{b}(w_n, w_n, -w) \to \hat{b}(w, w, -w) = 0$ as $n \to \infty$. By the definition of the mapping T we have

$$h\nu\|w_n - w\|^2 = \langle T(w_n), w_n - w\rangle_{E^*,E} - \left\langle w_n - \frac{\lambda}{\lambda + h} P_X w_n, w_n - w\right\rangle - h\nu\langle\langle w, w_n - w\rangle\rangle - h\hat{b}(w_n, w_n, w_n - w)$$

for $n \ge 1$, and the superior limit of the right-hand side is non-positive. This implies that the sequence $\{w_n\}$ converges in E to w as $n \to \infty$, and hence $\lim_{n\to\infty}\langle T(w_n), w_n - z\rangle_{E^*,E} = \langle T(w), w - z\rangle_{E^*,E}$ for any $z \in E$. This proves that T is pseudo-monotone. From [15] we conclude that $R(T) = E^*$, and the proof of the first assertion is completed. Setting $\phi = \bar{v}$ in (3.19), we have $|\bar{v}|^2 - |\bar{u}|^2 + |\bar{v} - \bar{u}|^2 + 2h\nu\|\bar{v}\|^2 = 0$ by (3.1), which implies that $|\bar{v}| \le |\bar{u}|$. Since $\|P_X\| \le 1$, we have $|\bar{u}| \le \frac{h}{\lambda+h}|f| + \frac{\lambda}{\lambda+h}|\bar{v}| \le \frac{h}{\lambda+h}|f| + \frac{\lambda}{\lambda+h}|\bar{u}|$. Hence $|\bar{u}| \le |f|$. □

To check condition (C2), let $u \in D(A)$. Let $\lambda_0 > 0$ be a number as in Lemma 3.4. Since $|u|^2 \le r$, we choose a smaller number λ_0 again such that $|u - \lambda_0 Au|^2 \le r_0$. Let $f = u - \lambda_0 Au$. Then we have $f \in D_h$ for $h > 0$. From Lemma 3.5 we deduce that for $h > 0$ there exist $u^h \in X$ and $v^h \in \boldsymbol{H}_0^1(\Omega)$ such that

$$\langle(v^h - u^h)/h, \phi\rangle + \nu\langle\langle v^h, \phi\rangle\rangle + \hat{b}(v^h, v^h, \phi) = 0 \quad \text{for } \phi \in \boldsymbol{H}_0^1(\Omega), \tag{3.21}$$

$$u^h = \frac{h}{\lambda_0 + h} f + \frac{\lambda_0}{\lambda_0 + h} P_X v^h. \tag{3.22}$$

Moreover, we have $|v^h| \le |u^h| \le |f|$. In particular, we have $u_h \in D_h$ for $h > 0$. Since u^h satisfies (3.21) and (3.22), we have $C_h u^h \ni \frac{\lambda_0+h}{\lambda_0}\left(u^h - \frac{h}{\lambda_0+h} f\right) = u^h + (h/\lambda_0)(u^h - f)$ by the definition of C_h. Hence $(u^h - f)/\lambda_0 \in A_h u^h$. Once it is proved that $u^h \to u$ in $\boldsymbol{L}^2(\Omega)$ as $h \to 0+$, condition (C2) is verified since $A_h u^h \ni (u^h - f)/\lambda_0 \to (u - f)/\lambda_0 = Au$ as $h \to 0+$ and since $\varphi_h(u^h) \le \exp(\nu^{-1}c_\nu(1 + r_0)|u^h|^2)|(u^h - f)/\lambda_0|^2$ and the right-hand side is bounded as $h \to 0+$.

Now, we shall show that $u^h \to u$ in $\boldsymbol{L}^2(\Omega)$ as $h \to 0+$. For this purpose, let $\{h_n\}$ be any null sequence of positive numbers. Since $(u - f)/\lambda_0 = Au$, we deduce from [14, Proposition 1.1.1 and Remark 1.1.4] that there exists $p \in L^2(\Omega)$ such that

$$\langle(u - f)/\lambda_0, \phi\rangle + \nu\langle\langle u, \phi\rangle\rangle + b(u, u, \phi) + \langle p, \operatorname{div}\phi\rangle = 0 \quad \text{for } \phi \in \boldsymbol{H}_0^1(\Omega). \tag{3.23}$$

We notice that $b(u, u, \phi) = \hat{b}(u, u, \phi)$ for $\phi \in \boldsymbol{H}_0^1(\Omega)$, since $u \in V$ and

$$b(v, w, z) + b(v, z, w) = 0 \quad \text{for } v \in V \text{ and } w, z \in \boldsymbol{H}_0^1(\Omega). \tag{3.24}$$

Subtracting (3.23) from (3.21) and setting $\phi = v^h - u \in \boldsymbol{H}_0^1(\Omega)$ in the resulting equality, we have

$$\begin{aligned}&\langle((v^h - u) - (u^h - u))/h + ((v^h - u) - (v^h - f))/\lambda_0, v^h - u\rangle + \nu\|v^h - u\|^2\\ &\quad + (\hat{b}(v^h, v^h, v^h - u) - \hat{b}(u, u, v^h - u)) = \langle p, \operatorname{div}(v^h - u)\rangle.\end{aligned}$$

By (3.1) we have $\hat{b}(v^h, v^h, v^h - u) = \hat{b}(v^h, u, v^h - u)$, and so we find that

$$\begin{aligned}&(|v^h - u|^2 - |u^h - u|^2)/h + (|v^h - u|^2 - |v^h - f|^2)/\lambda_0 + 2\nu\|v^h - u\|^2\\ &\quad \le 2\langle p, \operatorname{div}(v^h - u)\rangle - 2\hat{b}(v^h - u, u, v^h - u)\\ &\quad \le \epsilon|\operatorname{div}(v^h - u)|^2 + c_\epsilon|p|^2 + \epsilon\|v^h - u\|^2 + c_\epsilon(1 + |u|^2)\|u\|^2|v^h - u|^2\end{aligned}$$

for $\epsilon > 0$, where we have used (3.12) to obtain the last inequality. Thus, we have

$$\begin{aligned}&(|v^h - u|^2 - |u^h - u|^2)/h + (|v^h - u|^2 - |v^h - f|^2)/\lambda_0 + \nu\|v^h - u\|^2\\ &\quad \le C(\nu)|p|^2 + C(\nu)(1 + |u|^2)\|u\|^2|v^h - u|^2\end{aligned} \tag{3.25}$$

for some constant $C(\nu) > 0$. Since $u = P_X u$ and $\|P_X\| \le 1$, by (3.22) we have $|u^h - u|^2 \le \frac{h}{\lambda_0+h}|f - u|^2 + \frac{\lambda_0}{\lambda_0+h}|v^h - u|^2$. Substituting this into (3.25) we have

$$\lambda_0\nu\|v^h - u\|^2 \le |f - u|^2 + |v^h - f|^2 + \lambda_0 C(\nu)|p|^2 + \lambda_0 C(\nu)(1 + |u|^2)\|u\|^2|v^h - u|^2.$$

This implies that the sequence $\{v^{h_n}\}$ is bounded in $\boldsymbol{H}_0^1(\Omega)$. Let $u_n = u^{h_n}$ and $v_n = v^{h_n}$ for $n \ge 1$. Since $\boldsymbol{H}_0^1(\Omega)$ is compactly embedded in $\boldsymbol{L}^2(\Omega)$, there exist $\hat{u} \in \boldsymbol{H}_0^1(\Omega)$ and a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $v_{n_k} \to \hat{u}$ in $\boldsymbol{L}^2(\Omega)$ and $v_{n_k} \to \hat{u}$ weakly in $\boldsymbol{H}_0^1(\Omega)$ as $k \to \infty$. By (3.22) we have $u_{n_k} = \frac{h_{n_k}}{\lambda_0+h_{n_k}}f + \frac{\lambda_0}{\lambda_0+h_{n_k}}P_X v_{n_k} \to P_X\hat{u}$ as $k \to \infty$. Setting $\phi = v^h \in \boldsymbol{H}_0^1(\Omega)$ in (3.21) we have $(|v^h|^2 - |u^h|^2) + |v^h - u^h|^2 + 2\nu h\|v^h\|^2 = 0$, from which it follows that $|v_{n_k} - u_{n_k}|^2 \le |u_{n_k}|^2 - |v_{n_k}|^2 \to |P_X\hat{u}|^2 - |\hat{u}|^2 \le 0$ as $k \to \infty$. Thus, we have $\hat{u} = P_X\hat{u}$. Since $\boldsymbol{H}_0^1(\Omega) \cap X = V$, we have $\hat{u} \in V$. By (3.24), this implies that $\hat{b}(\hat{u}, \hat{u}, \phi) = b(\hat{u}, \hat{u}, \phi)$ for $\phi \in V$. By (3.22) we have $\langle(\frac{\lambda_0}{\lambda_0+h})^{-1}(u^h - \frac{h}{\lambda_0+h}f), \phi\rangle = \langle v^h, \phi\rangle$ for $\phi \in X$, which is rewritten as $\langle(u^h - f)/\lambda_0, \phi\rangle = \langle(v^h - u^h)/h, \phi\rangle$ for $\phi \in X$. Combining this and (3.21) we have $\langle(u^h - f)/\lambda_0, \phi\rangle + \nu\langle\langle v^h, \phi\rangle\rangle + \hat{b}(v^h, v^h, \phi) = 0$ for $\phi \in V$. Setting $h = h_{n_k}$ in this equality and letting $k \to \infty$ we have (3.18) by (3.3). From Lemma 3.4 we deduce that $u = \hat{u}$. Since for any null sequence $\{h_n\}$ of positive numbers, the sequence $\{u^{h_n}\}$ has a subsequence converging to the common element u in $\boldsymbol{L}^2(\Omega)$, we conclude that $u^h \to u$ in $\boldsymbol{L}^2(\Omega)$ as $h \to 0+$. Thus, condition (C2) is shown to be satisfied. Theorem 3.1 follows from Theorem 2.2.

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